Instability of a constrained pendulum system

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Linear perturbation analysis is used to determine the natural frequency of two pendulums connected by a rod. The analysis indicates a zone of instability in what looks like a stable system. The paradoxical phenomenon is explained, and a simple experiment confirms the instability. © 2011 American Association of Physics Teachers.

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I. INTRODUCTION

The simple pendulums shown in Fig. 1(a) are in static equilibrium. Under ideal conditions and perfect alignment they will remain stationary. If the lower pendulum $AB$ is slightly perturbed, it will oscillate with a small amplitude about its stable equilibrium position. If the upper inverted pendulum $A'B'$ is slightly perturbed, it will fall down. The upper pendulum is in unstable equilibrium.

Each one of the pendulums $ABCD$ and $A'B'C'D'$, illustrated in Fig. 1(b), consists of two equal point masses connected by three weightless rigid rods of equal length. They are confined to motion in a plane. If we use our intuition to determine which one of the pendulum systems is in stable equilibrium, we might err.

The formal way of determining the nature of stability of equilibrium is by linear stability analysis. For similar problems the analysis consists of the following steps: determine the equilibrium position, determine the nonlinear differential equations of motion, and determine the solution of the linearized differential equations of motion for small perturbations about the equilibrium. If the dynamics of the perturbed system is oscillatory, the equilibrium is stable. Otherwise, it is unstable. This process is demonstrated in Secs. II–IV for a system of two pendulums connected by a rod. The results in Sec. V reveal a counterintuitive result, in that the linear stability analysis indicates a zone of instability in what seems to be an inherently stable configuration. The paradoxical behavior is resolved in Sec. VI. An experiment described in Sec. VII confirms that the equilibrium position in the region indicated is unstable.

There is a wealth of literature concerning stability of an inverted pendulum, linearization of pendulum motion, energy of interaction, and dynamics of double pendulums. Stability analysis goes back to the work of Euler on buckling of columns. The book of Ziegler on structural stability is small in volume, wealthy in materials, and easy to read. Timoshenko and Gere is a classic reference on elastic instability. Perturbation methods for vibrating systems may be found in Meirovitch and Nayfeh. A chain of pendulums similar to the one that we study here could represent a discrete model of a vibrating cable. Static and dynamic analysis of cables is given by Irvine and Tadjbaksh and Wang. Chen, Li and Ro have conducted linear stability analysis of a heavy flexible beam that is resting on frictionless point supports. Their analysis is similar in nature to one that we do here.

II. EQUATIONS OF MOTION

The constrained double-pendulum system consists of two point masses $m$, pin-connected by three rigid weightless links of length $l$, as shown in Fig. 2(a). The distance between the pivots $O_A$ and $O_B$ is $\Delta$, the gravitational field is $g$, and the angles $\theta_k(t)$, where $k=1,2,3$, of the links from the vertices are defined in Fig. 2(a). For convenience we define a dimensionless distance $d=\Delta/l$. It is clear from Fig. 2(a) that $-1<d<3$; $d<0$ indicates that two links cross each other such that the pivot $O_B$ is left to the pivot $O_A$.

Figure 2(b) shows the system in equilibrium and the static angles $\vartheta_k$ of the links. We consider the in-plane motion of this system. The potential energy of the system is

$$V = mg(l \cos \vartheta_1 + \cos \vartheta_2 - 2 \cos \vartheta_1 - \cos \vartheta_2),$$

and the kinetic energy takes the form

$$T = \frac{1}{2} ml^2(2 \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 \dot{\theta}_1 \dot{\theta}_2 \cos(\vartheta_2 - \vartheta_1)).$$

The constraints are

$$\phi_1 = l (\sin \vartheta_1 + \sin \vartheta_2 + \sin \vartheta_3) - \Delta = 0, \quad (3)$$

$$\phi_2 = l (\cos \vartheta_1 + \cos \vartheta_2 + \cos \vartheta_3) = 0, \quad (4)$$

which imply, respectively, that the horizontal and vertical distances between $O_A$ and $O_B$ are $\Delta$ and zero. The associated Lagrangian is

$$L = T - V - \lambda mg \phi_1 - \mu mg \phi_2,$$

where $\lambda$ and $\mu$ are time dependent Lagrange multipliers. The motion of the system is determined by the Euler–Lagrange equations; see, for example, Langhaar: 12

$$\frac{\partial L}{\partial \dot{\vartheta}_k} - \frac{d}{dt} \frac{\partial L}{\partial \vartheta_k} = 0 \quad (k = 1,2,3),$$

which gives
Equations (1), (2), and (7) consist of five relations with five unknowns: \( \theta_k(t) \) (where \( k = 1, 2, 3 \)), \( \tilde{\mu}(t) \), and \( \tilde{\lambda}(t) \). They determine the dynamics of the three links and the two Lagrange multipliers.

### III. THE EQUATIONS OF STATIC EQUILIBRIUM

The equations of static equilibrium are obtained from their dynamic counterparts, Eqs. (7), (3), and (4), by setting the time derivatives equal to zero and substituting \( \theta_k \) for \( \theta_k \), \( \mu \) for \( \tilde{\mu} \), and \( \lambda \) for \( \tilde{\lambda} \),

\[
(2 - \mu) \sin \theta_1 + \lambda \cos \theta_1 = 0,
\]

\[
(1 - \mu) \sin \theta_2 + \lambda \cos \theta_2 = 0,
\]

\[
- \mu \sin \theta_3 + \lambda \cos \theta_3 = 0,
\]

\[
\sin \theta_1 + \sin \theta_2 + \sin \theta_3 = d,
\]

\[
\cos \theta_1 + \cos \theta_2 + \cos \theta_3 = 0.
\]

Equations (8) have the closed form solution,

\[
\theta_1 = \sin^{-1}\frac{d - 1}{2}, \quad \theta_2 = \frac{\pi}{2}, \quad \theta_3 = \pi - \theta_1, \quad \mu = 1, \quad \lambda = -\tan \theta_1.
\]

### IV. LINEARIZATION

To describe small oscillations about equilibrium we let

\[
\theta_k(t) = \theta_k + \epsilon_k(t), \quad k = 1, 2, 3,
\]

\[
\tilde{\lambda} = \lambda + \epsilon_\lambda(t), \quad \tilde{\mu} = \mu + \epsilon_\mu(t),
\]

where \( \epsilon_k \), \( \epsilon_\lambda \), and \( \epsilon_\mu \) are infinitesimal quantities. We substitute Eq. (10) into Eqs. (7), use Eqs. (8), and eliminate higher order terms in \( \epsilon \) to obtain

\[
l(2\ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_2 - \theta_1) - \dot{\theta}_2^2 \sin(\theta_2 - \theta_1))
- (\tilde{\mu} - 2)g \sin \theta_1 + \tilde{\lambda}g \cos \theta_1 = 0,
\]

\[
l(\ddot{\theta}_2 + \tilde{\mu} \sin(\theta_1 - \theta_2) - \ddot{\theta}_1 \sin(\theta_1 - \theta_2))
- (\tilde{\mu} - 1)g \sin \theta_2 + \tilde{\lambda}g \cos \theta_2 = 0,
\]

\[
- \tilde{\mu}g \sin \theta_3 + \tilde{\lambda}g \cos \theta_3 = 0.
\]

Linearization of the constraints gives

\[
\epsilon_1 \cos \theta_1 + \epsilon_2 \cos \theta_2 + \epsilon_3 \cos \theta_3 = 0.
\]
by virtue of Eqs. (3) and (4).
Equations (11)–(13) may be written in matrix form

$$\mathbf{M} \ddot{\mathbf{v}} + \mathbf{K} \mathbf{v} = \mathbf{0},$$

(14)

where

$$\mathbf{v} = (v_1, v_2, v_3, v_{\lambda}, v_{\mu})^T,$$

(15)

$$\mathbf{K} = \begin{bmatrix}
\kappa_{11} & 0 & 0 & \cos \theta_1 & -\sin \theta_1 \\
0 & \kappa_{22} & 0 & \cos \theta_2 & -\sin \theta_2 \\
0 & 0 & \kappa_{33} & \cos \theta_3 & -\sin \theta_3 \\
\cos \theta_1 & \cos \theta_2 & \cos \theta_3 & 0 & 0 \\
-\sin \theta_1 & -\sin \theta_2 & -\sin \theta_3 & 0 & 0
\end{bmatrix},$$

(16)

$$\kappa_{ii} = ((3 - i) - \mu) \cos \theta_i - \lambda \sin \theta_i \quad (i = 1, 2, 3),$$

(17)

$$\mathbf{M} = \begin{bmatrix}
2 & \cos(\theta_2 - \theta_1) & 0 & 0 & 0 \\
\cos(\theta_1 - \theta_2) & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

(18)

It follows from Eqs. (14)–(18) that, similar to the vibrations of a simple pendulum, the natural frequency of small oscillations in the double pendulum is independent of \(m\) and is proportional to \(\sqrt{g/l}\).

The solution of Eq. (14) takes the form

$$\mathbf{v}(t) = \mathbf{v} \sin \omega t,$$

(19)

where \(\mathbf{v}\) is a constant vector. Substitution of Eq. (19) into Eq. (14) gives a generalized eigenvalue problem,

$$(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{v} = \mathbf{0}.$$  

(20)

If we use Eq. (9), Eqs. (16)–(18) can be simplified to

$$\mathbf{K} = \begin{bmatrix}
1/\cos \theta_1 & 0 & 0 & \cos \theta_1 & -\sin \theta_1 \\
0 & 1/\tan \theta_1 & 0 & 0 & 1 \\
\cos \theta_1 & 0 & -\cos \theta_1 & 0 & 0 \\
-\sin \theta_1 & 1 & \sin \theta_1 & 0 & 0
\end{bmatrix},$$

(21)

$$\mathbf{M} = \begin{bmatrix}
2 & \sin \theta_1 & 0 & 0 & 0 \\
\sin \theta_1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$  

(22)

By equating the determinant of \(\mathbf{K} - \omega^2 \mathbf{M}\) to zero we find that Eq. (20), with \(\mathbf{K}\) and \(\mathbf{M}\) given by Eqs. (21) and (22), has one finite eigenvalue,

$$\omega^2 = \frac{1 + 2 \sin^2 \theta_1}{\cos \theta_1 g/l}.$$  

(23)

The corresponding eigenvector is

$$\mathbf{v} = \mathbf{e}_1 (1, -2 \sin \theta_1, 1, 1 + 2 \sin^3 \theta_1, \tan \theta_1 - 2 \cos \theta_1 \sin^2 \theta_1)^T.$$  

(24)

V. RESULTS

Figure 3 shows Eq. (23) for the frequency as a function of \(d\). If \(d = 1\), the slope of link 1 in the static position is \(\theta_1 = 0\). In this case Eq. (23) gives \(\omega^2 = g/l\). This result is expected because both pendulums, the one consisting of link 1 with its attached mass and the one consisting of link 3 with its attached mass, oscillate individually about vertical axes with a common natural frequency \(\omega = \sqrt{g/l}\). Link 2 synchronizes the motion of the pendulums, but leaves their period of oscillations unchanged.

If \(d = 0\) the slope of link 1 in the static position is \(\theta_1 = -\pi/6\). Equation (23) gives

$$\omega^2 = \frac{g}{l} \cos(\pi/6).$$  

(25)

In this case the double pendulum is reduced to the physical pendulum shown in Fig. 4. The natural frequency of a physical pendulum is (see Ref. 13, pp. 230–231)
For the physical pendulum of Fig. 4, that the system is unstable when the distance between the pivot and the center of gravity is slightly increased, the stable static position would be directly above or below the center of gravity, which means that the non-symmetric configuration is unstable, then when the distance between the pivots is slightly increased, the stable static position would be near the unstable configuration with horizontal link 2.

To examine this hypothesis we inspect the free-body-diagram for $d>1$ in Fig. 6(a). If the system is in equilibrium, then the sum of the moments of all external forces about any point should vanish. Let C be the intersection of the extensions of links 1 and 3, and let G be the center of gravity for the pendulum system, that is, G is the midpoint of link 2, as shown in Fig. 6(a). Because moment summation about C vanishes only when G is aligned vertically with C, it follows that a necessary condition for static equilibrium is that C lies directly above or below G. It thus follows that the only possible static equilibrium positions when $d>0$ are when link 2 is horizontal, that is, the lower configuration $P$ or the upper configuration $Q$ shown in Fig. 6(b). But if $d<d_c$, it is possible that there is an asymmetric static equilibrium. We solved Eq. (8) with $d=0.6$ and found the two neighboring solutions listed in Table I. The configurations associated with these solutions, labeled $P$ and $Q$, are illustrated in Fig. 7. The frequency plot in Fig. 3 corresponds to configuration $P$ in Fig. 7. The center of gravity $G_Q$ in configuration $Q$ is lower than $G_P$, which means that the non-symmetric configuration $Q$ is the stable one. It follows that when the pendulums are perturbed from configuration $P$, they move to stable configuration $Q$ and oscillate about it. The plot in Fig. 3 in the range $d<d_c$ indicates that the system has moved from configuration $P$ to $Q$. In summary, if $d=d_c$ the pendulum system is stable for nonzero perturbations. If the distance between the pivots $O_A$ and $O_B$ is increased beyond $d_c$, then the equilibrium configuration where link 2 is horizontal is no longer.

VI. THE PARADOX RESOLVED

Let us accept the results of the linear perturbation analysis and assume that when $d<d_c$ the static configuration of the pendulum system with horizontal link 2 is unstable. A small perturbation will cause the system to move to a stable configuration and to oscillate about it. If our intuition regarding stability did not mislead us, and the system is not strongly unstable, then when the distance between the pivots $O_A$ and $O_B$ is slightly increased, the stable static position would be near the unstable configuration with horizontal link 2.

To examine this hypothesis we inspect the free-body-diagram for $d>1$ in Fig. 6(a). If the system is in equilibrium, then the sum of the moments of all external forces about any point should vanish. Let C be the intersection of the extensions of links 1 and 3, and let G be the center of gravity for the pendulum system, that is, G is the midpoint of link 2, as shown in Fig. 6(a). Because moment summation about C vanishes only when G is aligned vertically with C, it follows that a necessary condition for static equilibrium is that C lies directly above or below G. It thus follows that the only possible static equilibrium positions when $d>0$ are when link 2 is horizontal, that is, the lower configuration $P$ or the upper configuration $Q$ shown in Fig. 6(b). But if $d<d_c$, it is possible that there is an asymmetric static equilibrium. We solved Eq. (8) with $d=0.6$ and found the two neighboring solutions listed in Table I. The configurations associated with these solutions, labeled $P$ and $Q$, are illustrated in Fig. 7. The frequency plot in Fig. 3 corresponds to configuration $P$ in Fig. 7. The center of gravity $G_Q$ in configuration $Q$ is lower than $G_P$, which means that the non-symmetric configuration $Q$ is the stable one. It follows that when the pendulums are perturbed from configuration $P$, they move to stable configuration $Q$ and oscillate about it. The plot in Fig. 3 in the range $d<d_c$ indicates that the system has moved from configuration $P$ to $Q$. In summary, if $d=d_c$ the pendulum system is stable for nonzero perturbations. If the distance between the pivots $O_A$ and $O_B$ is increased beyond $d_c$, then the equilibrium configuration where link 2 is horizontal is no longer.

Table I. Stable and unstable configurations for $d=0.6$.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>$\delta_1$ (deg)</th>
<th>$\delta_2$ (deg)</th>
<th>$\delta_3$ (deg)</th>
<th>$\lambda$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stable configuration $Q$</td>
<td>$-62.78$</td>
<td>$73.40$</td>
<td>$-138.00$</td>
<td>$1.2306$</td>
<td>$1.3669$</td>
</tr>
<tr>
<td>Unstable configuration $P$</td>
<td>$-53.13$</td>
<td>$90$</td>
<td>$-126.86$</td>
<td>$1.3333$</td>
<td>$1.0000$</td>
</tr>
</tbody>
</table>
stables. If \( d \) is slightly smaller than \( d_c \), the effect of the instability is that the pendulums oscillate about a slightly perturbed asymmetric configuration. Inspection of Fig. 7 and Table I suggests that the instability is substantial. A change in the distance between the pivots of \( d - d_{CR} = -0.0126 \) causes a shift of 16.6° in the stable static equilibrium position of link 2.

VII. EXPERIMENT

An experiment was conducted to verify the results. Two small holes, a distance \( l = 305 \text{ mm} \) apart, were made in a steel bar with dimensions of \( 320 \times 18 \times 4 \text{ mm}^3 \). Two loops of non-extensible strings of negligible mass, each of effective length \( l \), were inserted into the holes of the bar at one end and in small hooks that are mounted on a smooth board at the other end, as shown in Fig. 8. The board confines the steel bar to be in an in-plane configuration. The board is smooth to reduce frictional effects which were not taken into account in the model.

It might be argued that the bar-string system used in the experiment differs from the constrained double-pendulum model. However, the two systems are statically equivalent, and their dynamics for small perturbations are related to each other by a scale factor. Suppose that the mass of the steel bar is \( M = 2m \) and its moment of inertia about its center of gravity is \( I_G \). Then, when the bar-string system is perturbed from equilibrium, its total potential energy is

\[
V = 2mgh, \tag{29}
\]

where \( h \) is the vertical elevation of the center of gravity. Equation (29) also describes the total potential energy of the constrained double-pendulum system when it undergoes the same perturbation as the bar-string system. In addition, the two systems have the same geometrical constraints given in Eqs. (3) and (4). Because the equations of equilibrium are determined by minimization of the potential energy subject to the constraints, we conclude that the equations of equilibrium for the two systems are the same. Consequently the two systems have the same static equilibrium configurations.

Many problems associated with vibrations of a bar are formulated in Ref. 14.

When the pendulum system is slightly perturbed, its kinetic energy is

\[
T_P = m u^2 + \frac{m l^2 \dot{\theta}^2}{4}, \tag{30}
\]

where \( v \) is the linear velocity of the center of gravity of the pendulum system, that is, the midpoint of link 2, and \( \dot{\theta} \) is the angular velocity of link 2. The quantity \( v \) is related to \( \dot{\theta} \) by a scale factor due to the constraints. Let \( \dot{\theta} = \alpha \dot{\theta} \), where \( \alpha \) is the proportionality constant. Then the kinetic energy for the pendulum system is

\[
T_P = m \left( \alpha^2 + \frac{l^2}{4} \right) \dot{\theta}^2. \tag{31}
\]

Similarly, the kinetic energy for the bar-string system is

\[
T_B = m \left( \alpha^2 + \frac{I_G}{2m} \right) \dot{\theta}^2. \tag{32}
\]

Because the dynamic equations are obtained by minimization of the action, \( V - T \), subject to the constraints, and because \( I_G < ml^2/2 \), the period of oscillations of the pendulum system is larger than that of the bar-string system, but in all other aspects the two systems are identical dynamically. We thus expect that the bar-string and the double-pendulum systems have the same stable or unstable equilibrium states.

When the distance between the pivots is \( |\Delta| = 165 \text{ mm} \), the steel bar is approximately horizontal, as shown in Fig. 8(a). The small deviation from the horizontal position is attributed to frictional effects and imperfections in the string’s length and leveling of the pegs. This case, where \( |\Delta| = 165 \text{ mm} \) and \( l = 360 \text{ mm} \), corresponds to \( |d| = 0.541 < |d_c| = 0.587 \). The result is in agreement with the theory. The horizontal configuration of the bar is stable.

For \( |\Delta| = 210 \text{ mm} \) the steel bar tilted at an angle of about 40° with the horizon and remained in an asymmetric equilibrium, as shown in Fig. 8(b). This case corresponds to \( |d| = 0.688 > |d_c| = 0.587 \). The counterintuitive result predicted by the theory has thus been confirmed. The horizontal configuration of the bar is unstable.
VIII. CONCLUDING REMARKS

A double-pendulum system, configured in the manner shown in Fig. 5, can be unstable. Small perturbations can shift the system to an asymmetric state of equilibrium. In Sec. I we asked which of the configurations $ABCD$ or $A'B'C'D'$ is stable. The analysis, which was confirmed by an experiment, indicated that the correct answer is neither. They are both unstable configurations.

In the experiment we used a smooth board to confine the equilibrium configuration of the bar-string system to a plane. The bar tends to lower its center of gravity as much as possible. Without the board support the bar would rotate by $180^\circ$, untwist the cross arrangement of the strings, and reach the point of lowest center of gravity where the bar is horizontal.

The phenomenon that increasing the distance between two points beyond a certain limit in one direction causes a sudden large tilt of links can be used in designing mechanical switches and sensing devices. These devices could be activated when the strain between two points exceeds a maximum allowed level.