The Planck mass and the Chandrasekhar limit

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The Planck mass is often assumed to play a role only at the extremely high energy scales where quantum gravity becomes important. However, this mass plays a role in any physical system that involves gravity, quantum mechanics, and relativity. We examine the role of the Planck mass in determining the maximum mass of white dwarf stars. © 2009 American Association of Physics Teachers.

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I. INTRODUCTION

The Planck mass, length, and time can be constructed from the Newton gravitational constant \(G\), the Planck constant \(\hbar\), and the speed of light \(c\). In particular, the Planck mass, \(M_P\), is given by

\[
M_P = \sqrt{\frac{\hbar c}{G}}.
\]

The Planck mass is very small on the human scale because \(M_P \approx 10^{-8}\) kg. It is quite large on the scale of elementary particles because \(M_P/m_H \approx 10^{19}\), where \(m_H\) is the mass of a hydrogen atom. This large ratio is often referred to as the “hierarchy problem;” the “problem” is that such a large ratio seems unnatural and thus requires an explanation. Because present day particle accelerators probe energies corresponding to a few hundred \(m_Hc^2\), present day accelerators are very far from the much higher scale of \(M_Pc^2\), the energy scale at which quantum gravity becomes important.

One might expect that the Planck mass occurs only in very exotic, highly speculative, and completely theoretical areas of physics. However, because the Planck mass is constructed from only \(G\), \(\hbar\), and \(c\), it occurs in cases where gravity, quantum mechanics, and relativity all play an essential role. One such problem is the Chandrasekhar limit, the maximum mass of a white dwarf star. All stars are held in equilibrium by the opposing forces of self-gravity and pressure. Thus gravity plays a role in the properties of stars. In white dwarf stars, the pressure is supplied by the motion of electrons due to the uncertainty principle and the Pauli exclusion principle. Thus quantum mechanics plays a role. In the most extreme white dwarf stars, those nearing the maximum possible mass, the electrons are moving so fast that their speeds are comparable to the speed of light, and thus relativity plays an essential role.

In this paper we will give a simple derivation of the Chandrasekhar limit and show its connection with the Planck mass. We define \(N_C = M_P/m_H\) and call this large dimensionless number the hierarchy number. Nature also presents us with another large dimensionless number: the maximum number of nucleons (protons and neutrons) in a white dwarf star, which we will call the Chandrasekhar number and denote as \(N_C\). We can show that \(N_C = 0.774N_L\). We will show how this result comes about and specifically the roles of \(G\), \(\hbar\), and \(c\) in the calculation.

II. GRAVITATION

We start with some 19th century physics—the equilibrium configurations for spherical balls of self-gravitating gas. Every part of the gas is in equilibrium and thus must have zero net force on it. In particular, consider a small disk of the gas with base area \(A\) and thickness \(dr\) at a distance \(r\) from the center of the ball. Denote the mass of the disk by \(m_d\), and the mass of all the gas at radii smaller than \(r\) by \(M(r)\). Newton showed that the disk would be gravitationally attracted by the gas at radii smaller than \(r\), but unaffected by the gas at radii larger than \(r\), and the force on the disk would be the same as if all the mass at radii smaller than \(r\) were concentrated at the center. In other words, the inward force due to gravity on the disk has a magnitude

\[
F_{in} = \frac{GM(r)m_d}{r^2}.
\]

The volume of the disk is \(Adr\), and so we have \(m_d = \rho Adr\), where \(\rho\) is the density of the gas at radius \(r\). Thus using Eq. (2) we find

\[
F_{in} = \frac{GM(r)\rho Adr}{r^2}.
\]

If gravity is pulling the disk inward, then why does it not fall and the whole gas ball collapse? Because pressure differences push the disk outward. Let \(P(r)\) be the pressure of the gas as a function of radius. Then the outward force exerted on the inner surface of the disk is \(P(r)A\). This force is almost but not exactly balanced by the inward force of \(P(r+dr)A\). The net outward force on the disk due to pressure is

\[
F_{out} = P(r)A - P(r+dr)A = -A \frac{dP}{dr} dr.
\]

Because the inward force and outward force must cancel and thus must have equal magnitudes, it follows that \(F_{in} = F_{out}\), and therefore

\[
\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2}.
\]

As we increase \(r\) to \(r+dr\), \(M(r)\) increases because the mass contained within that radius is augmented by a thin shell of radius \(r\) and thickness \(dr\). Such a shell has volume \(4\pi r^2 dr\), and because the density at that radius is \(\rho\), it follows that \(M(r+dr) - M(r) = 4\pi r^2 \rho dr\), or

\[
\frac{dM(r)}{dr} = 4\pi r^2 \rho.
\]

Suppose that the matter that makes up the star is so simple that the pressure is a function only of the density (and not the temperature). If we know the equation of state \(P(\rho)\), we can rewrite Eq. (5) as
\[
\frac{dP}{dp} = \frac{GM(r)p}{r^2}. \tag{7}
\]

We then multiply Eq. (7) by \(r^2/(Gp)\), differentiate with respect to \(r\), and use Eq. (6) to obtain

\[
\frac{1}{4\pi G r^2} \left( \frac{\partial}{\partial r} \frac{dP}{dp} \right) + \rho = 0. \tag{8}
\]

Equation (8) allows us to make a mathematical model of the star. Choose a value of the central density \(\rho_c\) and integrate Eq. (8) outward until the pressure vanishes at the surface of the star. Complete the model by assuming vacuum at all radii larger than this stellar surface. Each value of \(\rho_c\) yields one stellar model.

### III. QUANTUM MECHANICS

Matter in a white dwarf consists of electrons and nuclei, mostly of carbon and oxygen. Because the matter is electrically neutral, there is one proton for every electron. Also because the number of neutrons equals the number of protons for both carbon and oxygen nuclei, there is one neutron for each electron. Usually the pressure of a gas depends on its temperature. However, the pressure at low temperatures is a purely quantum mechanical effect that is independent of its temperature. Recall that the uncertainty principle implies that particles confined to a box must have a momentum. Because faster moving particles have larger energy, it takes work to make the box smaller. This work is due to the pressure.

Consider \(N\) electrons along with \(N\) protons and \(N\) neutrons combined in carbon and oxygen nuclei confined to a cubical box of side \(L\). The electrons are free particles except for the confining walls of the box. Because of the Pauli exclusion principle, each state can have no more than two electrons. Thus for \(N\) electrons the ground state energy \(U\) is obtained by putting the \(N\) electrons in the \(N/2\) states of lowest energy. As shown in standard textbooks on statistical mechanics, the maximum momentum of an electron in the ground state \(p_{x}\) (which is the Fermi momentum) is given by

\[
p_{x} = (3\pi^2N)^{1/3} \frac{h}{L}. \tag{9}
\]

and the ground state energy is given by

\[
U = \frac{1}{\pi} \left( \frac{L}{h} \right)^3 \int_{0}^{p_{x}} \epsilon(p) p^2 dp, \tag{10}
\]

where \(\epsilon(p)\) is the energy of a single particle.

What we want to calculate is not \(U\) itself, but \(dP/dp\). Because \(P = -\partial U/\partial \mathcal{V}\), it follows that

\[
\frac{dP}{dp} = -\frac{\partial^2 U}{\partial \mathcal{V}}. \tag{11}
\]

(Here \(V = L^3\) is the volume of the box.) Recall that in addition to the \(N\) electrons the box contains \(N\) protons and \(N\) neutrons. Both the proton and the neutron have a mass approximately equal to \(m_H\) (the mass of a hydrogen atom) while the mass of the electron is much less. Hence the total mass of \(N\) protons, \(N\) neutrons, and \(N\) electrons is approximately \(2Nm_H\). Thus we find that \(\rho = 2Nm_H/V\) or

\[
\frac{N}{V} = \frac{\rho}{2m_H}. \tag{12}
\]

Hence, we can write Eq. (11) as

\[
\frac{dP}{dp} = \frac{V}{\rho} \frac{\partial U}{\partial \mathcal{V}}. \tag{13}
\]

From Eq. (9) it follows that \(p_{x} \approx V^{1/3}\) and therefore that

\[
\frac{\partial p_{x}}{\partial \mathcal{V}} = -\frac{1}{3} \frac{V}{p_{x}}. \tag{14}
\]

We apply Eq. (13) to Eq. (10) and use Eq. (14) and straightforward algebra and obtain

\[
\frac{dP}{dp} \approx \frac{6}{9\pi^2} \frac{\epsilon}{p^4} \left. \frac{d\epsilon}{dp} \right|_{p=p_{x}}. \tag{15}
\]

### IV. SPECIAL RELATIVITY

Equation (15) with \(p_{x}\) given by Eq. (9) gives \(dP/dp\), if we have an expression for \(\epsilon(p)\). In nonrelativistic mechanics \(\epsilon = p^2/(2m_e)\), where \(m_e\) denotes the mass of the electron. In relativistic mechanics

\[
\epsilon = \sqrt{p^2c^2 + m_e^2c^4}, \tag{16}
\]

and thus

\[
\frac{d\epsilon}{dp} = \frac{c^2p}{\sqrt{p^2c^2 + m_e^2c^4}}. \tag{17}
\]

We can gain insight by expressing Eq. (15) in terms of dimensionless quantities that have factored out the relevant physical scales. We define the dimensionless quantity \(X\) as

\[
X = \frac{p_{x}}{m_e c}. \tag{18}
\]

It follows from Eq. (9) that

\[
X^3 = \frac{\rho}{\tilde{\rho}}, \tag{19}
\]

where the constant \(\tilde{\rho}\) is given by

\[
\tilde{\rho} = \frac{2}{3\pi^2} \frac{m_H}{\lambda_e}, \tag{20}
\]

and \(\lambda_e\) is the Compton wavelength of the electron given by

\[
\lambda_e = \frac{\hbar}{m_e c}. \tag{21}
\]

It is straightforward to see the physical meaning of the scale \(\tilde{\rho}\): By the uncertainty principle, we expect electrons to acquire relativistic speeds at number densities on the order of one electron per cubic Compton wavelength. Because there are two nucleons per electron, this number density of electrons occurs at a mass density \(\rho\) that is of the order of \(m_H\) per cubic Compton wavelength. In other words, \(\tilde{\rho}\) is the mass density of white dwarf matter at which the electrons attain relativistic speeds.

We can use Eq. (17) in Eq. (15) to find

\[
\frac{dP}{dp} \approx \frac{1}{6} \frac{m_e c^2}{\sqrt{1 + X^2}}. \tag{22}
\]
V. PUTTING IT ALL TOGETHER

Given the expression for \(dP/d\rho\) we can construct stellar models. We use Eq. (22) in Eq. (8), and obtain

\[
\frac{3\pi}{16} \lambda^2 N^3 \rho \frac{d}{dr} \left( r^2 (1 + X^2)^{1/2} \frac{dX}{dr} \right) + X^3 = 0. \tag{23}
\]

Making a stellar model involves choosing a value \(X_c\) for \(X\) at the center of the star and then integrating Eq. (23) outward until the value of \(r\) at which \(X\) vanishes, which is the surface of the star. To streamline this integration we define \(Y = X/X_c\) so that \(Y = 1\) at the center of the star. We also introduce a dimensionless spatial coordinate in place of \(r\) as follows. Define the length \(a\) by

\[
a = \sqrt[3]{\frac{3\pi}{4X_c}}, \tag{24}
\]

and the dimensionless coordinate \(\zeta\) by

\[
\frac{\rho}{a} = \zeta. \tag{25}
\]

Then Eq. (23) becomes

\[
\frac{1}{2} \frac{d}{d\zeta} \left( \zeta^2 (1 + X^2)^{1/2} \frac{dX}{d\zeta} \right) + Y^3 = 0. \tag{26}
\]

Though not quite as intuitive as the density \(\bar{\rho}\), the physical meaning of the length scale \(a\) is clear. If we have white dwarf matter at mass density \(m_H/\lambda^3\), then because gravity is so weak, it must take a large amount of such matter for the gravity to be able to balance the enormous pressure. Thus there must be some large length scale such that the cube of that length scale is the volume needed to contain that large amount of matter at the given density. The expression for \(a\) tells us that the relevant length scale is of the order of \(N_h \lambda_c\).

Given a length scale and a density scale, we can combine them to make a mass scale. This mass scale must be of the order of \((m_H/\lambda^3)N_h^3 \lambda_c^3 = N^3_m m_H\). In other words, because the main contribution to the mass of the star is nucleons of mass \(m_H\), it follows that the relevant scale for the number of nucleons is \(N_h\).

What are these length and mass scales in more ordinary units? From the values of \(G, h, c, m_e\), and \(m_H\) we find that

\[
N_h = 1.300 \times 10^{19}, \tag{27}
\]

\[
\lambda_c = 3.863 \times 10^{-13} \text{ m}, \tag{28}
\]

and therefore

\[
N_h \lambda_c = 5.022 \times 10^6 \text{ m}, \tag{29}
\]

\[
N_h^3 m_H = 3.678 \times 10^{30} \text{ kg}. \tag{30}
\]

The radius \(R_\odot\) of Earth is \(R_\oplus = 6.378 \times 10^6 \text{ m}\), and the mass \(M_\odot\) of the Sun is

\[
M_\odot = 1.989 \times 10^{30} \text{ kg}. \tag{31}
\]

Hence, the mass and length scales of a white dwarf are

\[
N_h \lambda_c = 0.7874 R_\oplus, \tag{32}
\]

\[
N_h^3 m_H = 1.849 M_\odot. \tag{33}
\]

That is, the mass scale is of the same order as the mass of the Sun, and the length scale is of the same order as the radius of the Earth.

We can make the notion of length and mass scales more precise by asking how given a solution of Eq. (26) we would use it to calculate the radius and mass of the star. The solution \(Y(\zeta)\) vanishes at some \(\zeta_0\), which allows us to immediately calculate the radius \(R\) of the star as

\[
R = a \zeta_0. \tag{34}
\]

The mass of the star is given by integrating Eq. (6):

\[
M = \int_0^R 4\pi \rho r^2 dr, \tag{35}
\]

or

\[
M = \int_0^{\zeta_0} 4\pi \rho X_c^3 Y^3 a^3 \zeta^2 d\zeta, \tag{36}
\]

which using Eqs. (20) and (24) becomes

\[
M = \frac{\sqrt[3]{3\pi}}{8} N_h^3 m_H \int_0^{\zeta_0} Y^3 \zeta^2 d\zeta. \tag{37}
\]

This discussion of length and mass scales does not actually give us the sizes and masses of any white dwarf star. To do that we must solve Eq. (26) and use the results in Eqs. (34) and (37). Equation (26) cannot be solved in closed form, but it is easy to treat numerically using, for example, the fourth-order Runge–Kutta method.\(^4\) For each value of \(X_c\) we integrate Eq. (26) outward to find the value \(\zeta_0\) at which \(Y\) vanishes and then use Eqs. (34) and (37) to find the mass and radius of the star. Thus each \(X_c\) yields a point on a plot of \(R\) versus \(M\) for white dwarf stars. The result of such a calculation is shown in Fig. 1 (see also Ref. 5). Here \(R\) is measured in units of \(R_\oplus\), and \(M\) is measured in units of \(M_\odot\). It is clear from Fig. 1 that there is a limiting mass of about 1.43\(M_\odot\). What is not clear from the figure is why such a limiting mass exists. An answer can be found by an examination of Eq. (26). As the central density is increased without bound, Eq. (26) has a limit given by
\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{dY}{d\xi} \right) + Y^3 = 0.
\] (38)

To find the limiting mass, the Chandrasekhar mass \( M_C \), the mass in the limit of large central density, we need to numerically solve Eq. (38) and use the resulting \( Y(\xi) \) and \( \xi_0 \) in Eq. (37). The result is that

\[
M_C = 0.774N_C^2m_H.
\] (39)

or the limiting number of nucleons is \( N_C = 0.774N_h^3 \). We use the conversion factor of Eq. (31) to find

\[
M_C = 1.43M_\odot.
\] (40)

To understand why this limiting mass exists we return to the central fact of stellar structure: stars are held in equilibrium by the opposing forces of gravity and pressure. Because compressing a star raises its pressure, a star will be compressed until its pressure is large enough to balance its self-gravity. Because gravity is a \( 1/r^2 \) force, compressing a star also increases the force of its self-gravity. Thus it is not obvious that an equilibrium will exist. In particular, once the electrons in a white dwarf star become highly relativistic, it turns out that as the star is compressed, the forces of self-gravity and pressure increase at the same rate. Thus the limiting mass can be compressed to arbitrarily high densities without ever reaching equilibrium.

This argument can be turned into an estimate of the limiting mass, as was first done by Landau. We can somewhat crudely approximate the white dwarf star as a uniform density sphere of radius \( R \) containing \( N \) electrons and \( 2N \) nucleons. The total energy of the star consists of a kinetic part due to the quantum mechanical motion of the electrons and a potential part due to the mutual gravitational attraction of the nucleons. (The kinetic part is what we referred to earlier as the ground state energy of the electrons.) The equilibrium configuration can be thought of as the star adjusting its radius to attain the lowest total energy. Because the mass is \( 2Nm_H \), a standard result of Newtonian gravity yields a potential energy of approximately \(-GN^2m_H^2/R\). The kinetic energy is given by Eq. (10). In the nonrelativistic limit, \( \epsilon = \hbar^2 \xi^2/(2m_\rho) \), we obtain \( U \sim \hbar^2N^{5/3}/(m_\rho R^2) \). Thus, as long as the electrons remain nonrelativistic, there is always an equilibrium because as the star shrinks the kinetic energy grows faster (as \( R^{-2} \)) than the potential energy, which grows as \( R^{-1} \). However, in the ultrarelativistic limit, \( \epsilon = pc \), Eq. (10) yields \( U \sim \hbar cN^{4/3}/R \). Thus the kinetic energy and potential energy grow at the same rate as the star shrinks. The energy can be made arbitrarily large and negative when the coefficient of \( 1/R \) in the potential term is larger than the coefficient of \( 1/R \) in the kinetic term, so the limiting mass is estimated to occur when these two coefficients are equal, that is, when

\[
GN^2m_H^2 = \hbar cN^{4/3},
\]

which yields \( N \sim N_h^3 \) and thus \( M_C \sim N_h^3m_H \).

We now consider the nature of the calculation that led to this limiting mass, and in particular the roles of \( G, \hbar, \) and \( c \). Although the end result depends on all three fundamental constants, it does so in a somewhat compartmentalized way. The derivation of Eq. (8) uses only Newtonian gravity and depends only on \( G \). Equation (15) is derived using only quantum mechanics and depends only on \( h \). Finally Eq. (17) is found using only special relativity and depends only on \( c \). At first sight, the compartmentalized nature of the calculation seems strange: if the electrons are moving at speeds comparable to the speed of light, we would expect to need relativistic quantum mechanics to describe them, yet the derivation of Eq. (15) seems to use only nonrelativistic quantum mechanics. (Eddington did raise an objection of this sort to Chandrasekhar’s result.) The only part of quantum mechanics (other than the Pauli exclusion principle) used in deriving Eq. (15) is that free particles with momentum \( \vec{p} \) have the wave function \( \exp[i\vec{p} \cdot \vec{x}/\hbar] \). This result holds in both nonrelativistic and relativistic quantum mechanical systems. One might also worry that the electrons are highly relativistic, but standard nonrelativistic fluid mechanics is used in the derivation of Eq. (8). What makes this treatment permissible is that the nucleons are much more massive than electrons, so that even when the electrons are moving relativistically, the protons and neutrons, which make up most of the mass of the fluid, are not. Thus the Chandrasekhar limit is one of the simplest uses of the Planck mass in that it involves all three fundamental constants, and yet can be treated in a way that involves only Newtonian gravity, quantum mechanics, and special relativity.

Finally, we consider whether the Chandrasekhar limit can be attained. Recall that the only physics used in the derivation of the Chandrasekhar limit (other than gravity) is the quantum mechanical properties of the electrons. Because the approach to the Chandrasekhar limit involves arbitrarily high density, we might expect that other physical effects will become important as this limit is approached. And that turns out to be the case. The two important effects are nuclear fusion and electron capture. White dwarf stars form because low mass stars become hot and dense enough to fuse hydrogen into helium and then helium into carbon and oxygen. But they do not become hot and dense enough to fuse carbon or oxygen. If a white dwarf accretes material from another star and if that extra material brings the white dwarf close to the Chandrasekhar limit, then the white dwarf becomes dense enough for fusion of carbon and oxygen to take place. What results is not a white dwarf at the Chandrasekhar limit, but a thermonuclear explosion that destroys the entire star. Such an explosion is called a type Ia supernova. Electron capture becomes important in another type of explosion: a type II supernova, which is brought about by the collapse of the iron core of a high mass star. In high mass stars nuclear fusion in the core proceeds all the way to iron, which is the most tightly bound nucleus. The iron core is supported by degeneracy pressure in much the same way as is a white dwarf star. As the core approaches the Chandrasekhar limit and its electrons become relativistic, conditions become favorable for electron capture, in which an electron and proton combine to form a neutron and a neutrino. Because a neutron has more mass than a proton and electron combined, electron capture does not take place unless something supplies extra energy. As the electrons in the stellar core become relativistic, their kinetic energy becomes sufficiently large for electron capture to proceed. Because electron capture removes electrons that were the source of the pressure holding up the core, electron capture leads to a catastrophic collapse of the core of the star. Note that the products of electron capture are neutrons and neutrinos. The core of neutrons collapses, but neutrons, like electrons obey the Pauli exclusion principle, and thus have a degeneracy pressure of their own. When the core becomes sufficiently dense, it is prevented from further collapse by neutron degeneracy pressure and becomes a neutron star. The neutrinos stream outward and cause the ejection of all the rest of the star in a huge explosion: a type II supernova. Note
that just as there is a maximum mass, the Chandrasekhar limit, that can be supported by electron degeneracy pressure, so there is a maximum mass for a neutron star. If the neutron star exceeds this mass, nothing will prevent its complete gravitational collapse, and it becomes a black hole. Thus we see that the Chandrasekhar limit cannot be attained, but rather its near conditions become sufficiently extreme to produce a catastrophic explosion.

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Cardboard Steam Locomotive Model. The Millington/Barnard Collection in the University of Mississippi Museum collection includes a unique set of three cardboard steam engine models by Salleron of Paris. This locomotive model still works properly, with the piston going back and forth, the driving wheel rotating and the slide valve mechanism operating. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)