Dyck path triangulations of products of simplices and extendability

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Outline of the talk

Triangulations of products of simplices

Dyck path triangulations and some relatives

Extendability of partial triangulations
**Outline of the talk**

**Triangulations of products of simplices**

- Dyck path triangulations and some relatives

- Extendability of partial triangulations
Products of simplices

The cartesian product of two standard simplices is the polytope

\[ \Delta_{d-1} \times \Delta_{n-1} := \text{conv} \left\{ (e_i, \bar{e}_j) : e_i \in \mathbb{R}^d, \bar{e}_j \in \mathbb{R}^n \right\} \subset \mathbb{R}^{d+n} \]

with \( d \cdot n \) vertices and of dimension \( d + n - 2 \).

For instance:
Triangulations of $\Delta_{d-1} \times \Delta_{n-1}$

A triangulation of $\Delta_{d-1} \times \Delta_{n-1}$ is a subdivision of $\Delta_{d-1} \times \Delta_{n-1}$ into subsimplices

( subsimplex = simplex spanned by vertices of $\Delta_{d-1} \times \Delta_{n-1}$ )

- that cover $\Delta_{d-1} \times \Delta_{n-1}$ and
- whose relative interiors don’t intersect.

For instance:
Triangulations of $\Delta_{d-1} \times \Delta_{n-1}$ elsewhere

Triangulations and coarser polyhedral subdivisions of $\Delta_{d-1} \times \Delta_{n-1}$ are interesting:

- **[Babson-Billera '98]**
  
  \[
  \begin{aligned}
  \text{coarse regular subdivisions of } & \Delta_{d-1} \times \Delta_{n-1} \\
  \leftrightarrow & \text{facets of Newton polytope of product of all minors of generic } d \times n \text{ matrix}
  \end{aligned}
  \]

- **[Sturmfels-Develin '04, Ardila-Develin '09 et al.]**
  
  \[
  \begin{aligned}
  \text{triangulations of } & \Delta_{d-1} \times \Delta_{n-1} \\
  \leftrightarrow & \text{generic arrangements of } d \text{ tropical hyperplanes in } \mathbb{T}P^{n-1}
  \end{aligned}
  \]

- **[Kapranov '92, Speyer '08, Herrmann-Joswig-Speyer '12]**
  
  \[
  \begin{aligned}
  \text{subdivisions of } & \Delta_{d-1} \times \Delta_{n-1} \\
  \leftrightarrow & \text{matroid polytope subdivisions of } d \text{-th hypersimplices of order } (n + d) \text{ at vertex figures}
  \end{aligned}
  \]

- **[Ardila-Billey '07, Ardila-Ceballos '11]**
  
  \[
  \begin{aligned}
  \text{triangulations of } & \Delta_{d-1} \times \Delta_{n-1} \\
  \leftrightarrow & \text{matroid of lines in generic arrangement of } n \text{ complete flags in } \mathbb{C}^d
  \end{aligned}
  \]
Some known properties of triangulations of $\Delta_{d-1} \times \Delta_{n-1}$

- All full-dimensional simplices of $\Delta_{d-1} \times \Delta_{n-1}$ have the same volume $\Rightarrow$ all triangulations of $\Delta_{d-1} \times \Delta_{n-1}$ have the same number of full-d’ simplices (namely $\binom{d+n-2}{d-1}$) [Folk-lore?].
- The secondary polytope of $\Delta_{d-1} \times \Delta_1$ is affinely isomorphic to $(d-1)$-permutahedron [Gelfand-Kapranov-Zelevinsky ’91].
- Arbitrarily large numbers are required to have an integral normal vector for certain facets of the secondary polytope of $\Delta_{d-1} \times \Delta_{n-1}$ [Babson-Billera ’98].
- The graphs of triangulations of $\Delta_{d-1} \times \Delta_1$ and of $\Delta_{d-1} \times \Delta_2$ are connected under flips [Santos ’04].
Triangulations of $\Delta_{d-1} \times \Delta_{n-1}$

Grid representation

Vertices of $\Delta_{d-1} \times \Delta_{n-1}$ can be represented in a $d \times n$ rectangular grid:

A (sub)simplex of $\Delta_{d-1} \times \Delta_{n-1}$ is a subset of the grid:

So a triangulation of $\Delta_{d-1} \times \Delta_{n-1}$ can look like:
Grid representation

- Consider a staircase in a $d \times n$ grid.
- The corresponding vertices of $\Delta_{d-1} \times \Delta_{n-1}$ span a $(n + d - 2)$-simplex.

- The $\left(\binom{d+n-2}{d-1}\right)$ simplices obtained from all staircases cover $\Delta_{d-1} \times \Delta_{n-1}$ and their interiors don’t intersect:

- They form the staircase triangulation of $\Delta_{d-1} \times \Delta_{n-1}$. 
Mixed subdivision representation

(Some) simplices of $\Delta_{d-1} \times \Delta_{n-1}$ can be represented as Minkowski sums inside a dilated simplex $d\Delta_{n-1} = \Delta_{n-1} + \ldots + \Delta_{n-1}$:

$$d \times \Delta_{n-1}$$

Theorem (Sturmfels ’94, Huber-Rambau-Santos ’00)

\[
\begin{align*}
\{ \text{triangulations of } \Delta_{d-1} \times \Delta_{n-1} \} & \quad \overset{\text{Cayley trick}}{\longleftrightarrow} \quad \{ \text{fine mixed subdivisions of } d\Delta_{n-1} \} \\
\end{align*}
\]
Mixed subdivision representation

Theorem (Develin-Sturmfels '04, Santos '04, Ardila-Develin '09, Oh-Yoo '12, Horn '12)

\[ \{ \text{fine mixed subdivisions of } d \Delta_{n-1} \} \leftrightarrow \{ \text{generic arrangements of } d \text{ tropical hyperplanes in } \mathbb{TP}^{n-1} \} \]

For the staircase triangulation, for instance:
Outline of the talk

Triangulations of products of simplices

Dyck path triangulations and some relatives

Extendability of partial triangulations
Dyck path triangulation of $\Delta_{n-1} \times \Delta_{n-1}$

Consider the Dyck paths in an $n \times n$ grid

with their orbits under $(i, j) \mapsto (i + 1 \mod n, j + 1 \mod n)$

Theorem (Ceballos-Padrol-S ’13)

The resulting $n \cdot \frac{1}{n} \binom{2(n-1)}{n-1}$ form a regular triangulation of $\Delta_{n-1} \times \Delta_{n-1}$: the Dyck path triangulation.
Dyck path triangulations and some relatives

Dyck path triangulation of $\Delta_{n-1} \times \Delta_{n-1}$

Mixed subdivision representation:

$\text{Mixed subdivision representation:}$

$\Delta_2 \times \Delta_2$

$4\Delta_3$

$\Delta_3 \times \Delta_3$
Some relatives

Theorem (Ceballos-Padrol-S ’13)
The following are all (regular) triangulations.

Flipped Dyck path triangulation:

Extended Dyck path triangulation:

“Rational” Dyck path triangulation
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Extendability of partial triangulations
Partial triangulations of $\Delta_{d-1} \times \Delta_{n-1}$

*It is an open problem due to Gel’fand, Kapranov and Zelevinsky to find an explicit description of all triangulations of $\Delta_{d-1} \times \Delta_{n-1}$. [Sturmfels ’91]*

**Our approach:** are there $d, n, k \in \mathbb{N}^+$ conditions s.t. every triangulation of $\text{ske}_{k-1}(\Delta_{d-1}) \times \Delta_{n-1}$ is the restriction of a triangulation of $\Delta_{d-1} \times \Delta_{n-1}$?

![Diagram](image.png)

- Extends
- Doesn't extend!
Partial triangulations of $\Delta_{d-1} \times \Delta_{n-1}$

**Our approach:** are there $d, n, k \in \mathbb{N}^+$ conditions s.t. every triangulation of

$$\text{skel}_{k-1}(\Delta_{d-1}) \times \Delta_{n-1}$$

is the restriction of a triangulation of $\Delta_{d-1} \times \Delta_{n-1}$?

**Motivation and existing results**

- $k = 2$, $\min\{d, n\} \leq 3$: one obstruction, complete characterization [Ardila-Ceballos ’11]
- $k = 2$, $\min\{d, n\} > 3$: more obstructions, open [Santos ’11, Ceballos-Padrol-S ’13]
- **Conjecture:** for $k = 2$, general $d, n$, there are $\infty$-many obstructions
- $d \geq n > k$: open. **Conjecture:** there are $\infty$-many obstructions
- $d \geq k \geq n$: solved [Ceballos-Padrol-S ’13]
Extendability result

Theorem (Ceballos-Padrol-S ’13)

Let $d \geq k > n \in \mathbb{N}$. Every triangulation of $\text{ske}_{k-1}(\Delta_{d-1}) \times \Delta_{n-1}$ extends to a unique triangulation of $\Delta_{d-1} \times \Delta_{n-1}$.

Some alternative interpretations

- “as $d$ increases, triangulations of $\Delta_{d-1} \times \Delta_{n-1}$ don’t get much more complicated than triangulations of $\Delta_{n} \times \Delta_{n-1}$”
- “when $d \gg n$, compatibly piecing together triangulations of $\Delta_{n} \times \Delta_{n-1}$ we can always build any triangulation of $\Delta_{d-1} \times \Delta_{n-1}$”
Extendability result “tropically”

Theorem (Ceballos-Padrol-S ‘13)

Let \( d \geq k > n \in \mathbb{N} \). Every triangulation of \( \text{skel}_{k-1}(\Delta_{d-1}) \times \Delta_{n-1} \) extends to a unique triangulation of \( \Delta_{d-1} \times \Delta_{n-1} \).

Tropically, when \( d > n \):

▸ a generic arrangement of \( d \) tropical pseudohyperplanes in \( \mathbb{T}\mathbb{P}^{n-1} \) gives rise to a “compatible collection” of \( \binom{d}{n+1} \) generic subarrangements of \( n+1 \) tropical pseudohyperplanes in \( \mathbb{T}\mathbb{P}^{n-1} \).

▸ conversely, every “compatible” collection of \( \binom{d}{n+1} \) generic subarrangements of \( n+1 \) tropical pseudohyperplanes in \( \mathbb{T}\mathbb{P}^{n-1} \) equals the collection of restrictions of a unique generic arrangement of \( d \) tropical pseudohyperplanes in \( \mathbb{T}\mathbb{P}^{n-1} \)

without adjective “tropical”, this is \[15/18\]
Non-extendability result

Notes

► $k \geq n$ already suffices to guarantee uniqueness
  if a triangulation of $\text{ske}_{k-1}(\Delta_{d-1}) \times \Delta_{n-1}$ extends, then extension is unique

► $k > n$ is necessary for existence, i.e., the bound is optimal

Theorem (Ceballos-Padrol-S ’13)

*For every natural number $n \geq 2$ there is a non-extendable triangulation of $\text{ske}_{n-1}(\Delta_n) \times \Delta_{n-1}$.***
Extended Dyck path triangulation of $\Delta_{n-1} \times \Delta_{n-1}$

**Theorem (Ceballos-Padrol-S '13):** The resulting triangulation of $\text{skel}_{n-1} \Delta_n \times \Delta_{n-1}$ is non-extendable.
Take away messages

- There is a triangulation of $\Delta_{n-1} \times \Delta_{n-1}$ using Dyck paths and cyclic symmetry.

- If $d \gg n$, any triangulation of $\Delta_{d-1} \times \Delta_{n-1}$ can be “built locally”, by piecing together triangulations of $\Delta_n \times \Delta_{n-1}$.

- For every $n \geq 2$ there are non-extendable triangulations of $\text{skel}_{n-1}(\Delta_n) \times \Delta_{n-1}$. 
Thank you!